Exercise 7

1. Let f be convex and $x_0 \in \text{int}(\text{dom}f)$. Prove that f is locally Lipschitz, i.e., $\exists \varepsilon > 0$ and $\bar{M}(x_0, \varepsilon) > 0$ such that

$$|f(y) - f(x_0)| \le \bar{M}(x_0, \varepsilon) \|x - y\|, \forall y \in B_\varepsilon(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \le \varepsilon\}.$$

2. Let f be a convex function defined on a subset of $\mathbb{R}$. Prove that f is globally Lipschitz in its its interior of domain. Namely, let $K \subset \mathbb{R}$ be a closed and bounded set contained in $\text{ri}(\text{dom} f)$, then there exists constant M such that

$$|f(x) - f(y)| \le M\|x - y\|_2 \quad \forall x, y \in K.$$

3. Explain why all the three assumptions on K in the second problem, i.e., (1) closedness, (2) boundedness, and (3) $K \subset \text{ri}(\text{dom} f)$ are essential.